Exponentially Weighted Moving Average (EWMA) Prediction in the Software Development Process

Thomas M. Fehlmann  
Euro Project Office AG  
Zurich, Switzerland  
eMail: thomas.fehlmann@e-p-o.com

Eberhard Kranich  
Euro Project Office  
Duisburg, Germany  
eMail: eberhard.kranich@e-p-o.com

Abstract—For some years, Statistical Process Controls (SPC) techniques such as traditional Shewhart control charts add value to monitor and to control the Software Development Process (SDP) efficiently. Nonetheless, the application of Shewhart control charts to the SDP involves a considerable problem, since the availability of a sufficiently large set of observations is essential when constructing traditional control charts. Especially at the start-up of each SDP phase such a set cannot be provided. To remedy this problem, Q control charts widely used when monitoring short-run manufacturing processes have been introduced successfully. This paper focuses on the predictive property of Exponentially Weighted Moving Average (EWMA) Q control charts and investigates whether the predictive property is attractive for monitoring and controlling the SDP. Results of initial experiments are also reported.

Keywords Q-Statistic, Q control chart, Exponentially Weighted Moving Average (EWMA), software development process (SDP), statistical process control (SPC), forecasting

I. INTRODUCTION

Shewhart control charts, introduced by Shewhart [34] in the 1920’s and further investigated by W. E. Deming [6] (see also Thompson and Koronački [36]), are a highly valuable and well accepted Statistical Process Control (SPC) tool for the monitoring, controlling, and systematic improvement of mass production processes manufacturing identical parts, see also standard SPC textbooks such as Montgomery [25], Qiu [26], and Ryan [33] for further details. Shewhart control charts are mainly applied to investigate and understand the variation of a considered process. In view of the fundamental equation of SPC, the variation of the considered process is equal to the natural cause process variation plus the assignable cause process variation. Natural or common cause variation is unpreventable inherent to any process and can only be reduced by modifying the process itself, whereas assignable or special cause variation originates from external process events which make the process in-stable or out-of-control and, when detected, should be eliminated promptly to stabilize the process.

The Software Development Process (SDP) is evidently not a mass production process, since an actually developed software artifact is in general not identical to other previously developed software artifacts. Hence, a sufficiently large set of observations is not available which is required for the construction of Shewhart control charts. However, Florac and Carleton [10] provide guidelines how to monitor and control the SDP by means of Shewhart control charts.

But how to monitor and control the SDP when only a small set of individual observations such as defects at the start-up stage of testing is available? Chang and Tong [2] were the first who apply short-run statistical process control techniques along with associated short-run control charts, also termed self-starting control charts, to the monitoring of the SDP, see also Fehlmann and Kranich [8].

In particular, Chang and Tong [2] recommend to utilize Q control charts introduced by Quesenberry [27], [30]. These control charts are based on standardized, normally distributed Q-Statistics. In contrast to classic Shewhart control charts, Q control charts feature the notable property that they have constant control limits. This property enables, for instance, a visualization of the monitored individual observations of different process characteristics such as Chillarege’s [3] Orthogonal Defect Classification (ODC) categorized defects in exactly one control chart.

In order to identify an out-of-control situation in the course of a process run, a set of decision rules is generally applied, see e.g. Hoyer and Ellis [17]. The main purpose of such rules is to signal a potential out-of-control situation before the actual in-control process reveals in fact an out-of-control event. Complementary to some of the decision rules or as an alternative to Shewhart control charts, Exponential Weighted Moving Average (EWMA) control charts which have individual observations as input can be utilized. Contrary to Shewhart control charts which can only detect large shifts in the observations, applying EWMA control charts is recommended when small shifts in the observations are to be identified, see e.g. Hunter [18], Montgomery [25], Qiu [26], or Ryan [33].

Quesenberry [28], [30] points out that the values of the various types of Q-Statistics can also serve as input to EWMA control charts resulting in EWMA Q control charts. This type of control charts is described in detail and investigated by Fehlmann and Kranich [7] in view of monitoring and controlling the software development process. The focus of this paper is to analyze whether the predictive property of EWMA Q control charts is an attractive control mechanism in the context of the software development process.

The paper is organized as follows. Basic principles of Q-Statistics and Q control charts are described in Section II. The background of EWMA control charts along with an enhancement is given in Section III. EWMA Q control charts are introduced in Section IV. Section V shows how EWMA Q control charts can be utilized to control the software development process by means of forecasting. The results of this section are applied to an example in Section VI in order to illustrate the attractiveness of forecasting future individual observations. Finally, conclusions are pointed out in Section VII.

II. Q-STATISTICS AND Q CONTROL CHARTS

In a sequence of papers Quesenberry [27], [28], [30] introduces normalized statistics, the Q-Statistics, in order to construct Shewhart type control charts for individual observations from, for instance, a normal distribution when the process parameters are unknown at the start-up of the considered process.

A. Q-Statistics from Individual Observations

Quesenberry [27], [28], [30] considers a sequence of first k independent, identically and normally distributed (i. i. d.) random variables
\{x_1, x_2, \ldots, x_k\} with mean \( \mu \) and variance \( \sigma^2 \), i.e., \( \{x_j\} \sim \mathcal{N}(\mu, \sigma^2) \) with \( 1 \leq j \leq k \). According to the parameters \( \mu \) and \( \sigma^2 \) Quesenberry [27], [28], [30] distinguishes four cases:

1. Case KK: \( \mu = \mu_0 \) and \( \sigma = \sigma_0 \) are known \( (k = 1, 2, \ldots) \)

\[
Q_k(x_k) = \frac{x_k - \mu_0}{\sigma_0}. \tag{1}
\]

2. Case UK: \( \mu \) is unknown, \( \sigma = \sigma_0 \) is known \( (k = 2, 3, \ldots) \)

\[
Q_k(x_k) = \sqrt{\frac{k-1}{k}} \left( \frac{x_k - \bar{x}_{k-1}}{\sigma_0} \right), \tag{2}
\]

with

\[
\bar{x}_k = \frac{1}{k} \sum_{j=1}^{k} x_j. \tag{3}
\]

3. Case KU: \( \mu = \mu_0 \) is known, \( \sigma \) is unknown \( (k = 2, 3, \ldots) \)

\[
Q_k(x_k) = \Phi^{-1} \left\{ G_{k-1} \left( \frac{x_k - \mu_0}{\sigma} \right) \right\}, \tag{4}
\]

where

\[
s_k^2 = \frac{1}{k-1} \sum_{j=1}^{k} (x_j - \bar{x}_k)^2. \tag{5}
\]

4. Case UU: \( \mu \) and \( \sigma \) unknown \( (k = 3, 4, \ldots) \)

\[
Q_k(x_k) = \Phi^{-1} \left\{ G_{k-2} \left[ \sqrt{\frac{k-1}{k}} \left( \frac{x_k - \bar{x}_{k-1}}{s_{k-1}} \right) \right] \right\}, \tag{6}
\]

where \( \bar{x}_{k-1} \) and \( s_{k-1} \) are defined in (3) and (5), respectively.

In (4) and in (6), \( G_{(j)} \) denotes the Student \( t \) cumulative distribution function with \( k-1 \) resp. \( k-2 \) degrees of freedom and \( \Phi^{-1} \) the inverse of the standard normal cumulative distribution function. For further details see Chang and Tong [2], Fehlmann and Kranich [8], Quesenberry [27], [30], Zantek [40], and Zantek and Nestler [41]. Obviously, Case UU is the most important and occurring case in practice and is therefore the only case investigated in this paper.

With respect to (6), Quesenberry [30] recommends to update \( \bar{x}_k \) and \( s_k^2 \) each time a new observation is available by means of the following sequential updating formulas instead of calculating \( \bar{x}_k \) and \( s_k^2 \) each time from scratch according to (3) and (5), respectively:

(a) The sample mean \( \bar{x}_k \) can be calculated sequentially by

\[
\bar{x}_k = \frac{1}{k} \sum_{j=1}^{k} x_k = \left( 1 - \frac{1}{k} \right) \bar{x}_{k-1} + \frac{1}{k} x_k \tag{7}
\]

for \( k \geq 2 \) and with \( \bar{x}_1 = x_1 \).

(b) The sample variance \( s_k^2 \) can be computed sequentially by

\[
s_k^2 = \frac{1}{k-1} \sum_{j=1}^{k} (x_j - \bar{x}_k)^2
= \left( 1 - \frac{2}{k-1} \right) s_{k-1}^2 + \frac{1}{k} (x_k - \bar{x}_{k-1})^2, \tag{8}
\]

for \( k \geq 3 \) and with

\[
s_1^2 = \frac{1}{2} (x_2 - \bar{x}_1)^2 = \frac{1}{2} (x_2 - x_1)^2. \]

Note that the formulas (7) and (8) are numerically more stable than the fundamental formulas (3) and (5), respectively.

\section*{B. Q Control Charts}

Quesenberry [27], and Zantek and Nestler [41] prove that each Q-Statistic \( Q_k(x_k) \) in (1), (2), (4), and (6), respectively, produce a sequence of independent \( \mathcal{N}(0, 1) \) distributed random variables. Consequently, the 3\( \sigma \) upper control limit (UCL), the center line (CL), and the 3\( \sigma \) lower control limit (LCL) of a Q control chart are fixed:

\[
UCL = +3, \quad CL = 0, \quad LCL = -3. \tag{9}
\]

It is well known that traditional Shewhart control charts help to decide whether a considered process is under statistical control or in-control by using certain run rules or tests. A set of such run rules are listed in e.g. Champ et al. [1], Florac and Carleton [10], Hoyer and Ellis [17], and Montgomery [25]. Quesenberry [28] applies a subset of that run rules to Q control charts. For instance, one such (simple) rule is the 1-of-1 or outlier test, i.e., the process signals an outlier observation, if \( Q_k(x_k) < LCL \) or \( Q_k(x_k) > UCL \). Quesenberry [29] proves that Q control charts are optimal to detect outliers.

The occurrence of an outlier generally requires some action, since an outlier may strongly impact the sequence of parameter estimates by masking the outlier effect when further observations are taken, compare (7) and (8), respectively. Excluding an outlier from subsequent parameter estimates the sensitivity to detect further outliers will be improved. According to Q control charts based on the Q-Statistic (6) an effective way to automatically build a “new” Q control chart from scratch is to eliminate the outlier and all previous observations from subsequent calculations of the parameter estimates. Based on the example of Florac and Carleton [10, pp. 151], Figure 1 and Figure 2 illustrate the observations sequence without and with eliminating outliers automatically.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Observations sequence without outlier elimination}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Observations sequence with outliers elimination}
\end{figure}

In Figure 1 and Figure 2 an outlier is marked at day 4, and in Figure 2 an additional outlier at day 13. In addition, Figure 2 illustrates that each time an outlier has been detected subsequent Q-Statistic data points are calculated from scratch by means of (6).
A benefit of applying $Q$ control charts is that the behavior of various process characteristics can be visualized in exactly one control chart, see Figure 3.

![Q Control Chart](image)

Fig. 3. $Q$ control chart of four process characteristics

Such a control chart gives more insight into the process behavior. For instance, in Figure 3 the peaks of the aggregated number of defects are obviously caused by out-of-control events of the further three considered defect types.

### III. EWMA CONTROL CHARTS

Exponentially Weighted Moving Average (EWMA) control charts are an alternative to traditional Shewhart type control charts, see, e.g. Hunter [18], Montgomery [25], Qiu [26], or Ryan [33]. EWMA defects are obviously caused by out-of-control events of the further control chart, see Figure 3. Obviously, a $\lambda \in (0, 1]$ is an appropriately chosen weighting factor. Montgomery [25] assigns more weight to the previous observation $x_k$ and less weight to $x_k$. Steiner [35] recommends to set $f = 1 + (1/k)$. Thus the adjusted EWMA control limits are of the form

$$ LCL = \mu_x - \rho \sigma_x \sqrt{\frac{\lambda}{2 - \lambda}}, \quad UCL = \mu_x + \rho \sigma_x \sqrt{\frac{\lambda}{2 - \lambda}}, $$

where $\rho > 0$ is chosen appropriately. Montgomery [25] points out that $(\lambda, \rho) \in \{ (0.05, 2.615), (0.10, 2.814), (0.20, 2.962), (0.25, 2.998) \}$ works well in practice, see likewise Lucas and Saccucci [24], and Qiu [26]. Evidently, a considered process is in-control, when $z_k$ in (10) is located between the EWMA control chart limits (16).

### B. An EWMA FIR Enhancement

In order to calculate the EWMA control chart limits in (16), the limiting variance given in (15) is often used in practice. A small $\lambda$ motivates this approach since the variance of $z_k$ in (13) slowly converges to its limiting variance (15) because of the slow convergence of $(1 - (1 - \lambda)^{2k}) \to 1$. Taking the limiting variance of $z_k$ impacts the start-up phase of a process since the sensitivity of an EWMA control chart to detect an out-of-control event in that phase gets lost.

The Fast Initial Response (FIR) feature compensates this problem, see e.g. Chiu [4], Haq, Brown and Molkchanova [13], Knoth [23], Rhoads, Montgomery and Mastrangelo [32], and Steiner [35]. The FIR feature narrows the EWMA control chart limits, at least in the course of the start-up phase of a process, and thereby increases the sensitivity of an EWMA control chart to detect an out-of-control event in that phase. The general FIR feature is defined by

$$ \text{FIR}_{adj} = \left(1 - (1 - f)^{1+\alpha(k-1)}\right)^b. $$

If $b = 1$, $\text{FIR}_{adj}$ in (17) reflects Steiner’s FIR adjustment [35], whereas the FIR adjustment of Haq, Brown and Molkchanova [13] results from setting $b = 1 + (1/k)$. Thus the adjusted EWMA control chart limits are of the form

$$ \mu_x \pm \rho \times \text{FIR}_{adj} \times \sigma_x \sqrt{\frac{\lambda}{2 - \lambda}}. $$

According to (17), the parameter $\alpha$ is chosen (or calculated) such that $\text{FIR}_{adj}$ has minor impact on the EWMA control chart limits (18) after a pre-defined (observation) index $k = k_0$, i.e., for indexes $k > k_0$ $\text{FIR}_{adj} \approx 1$ is required. For $k = 1$, the parameter $f \in (0, 1]$ reflects the proportion of the distance the FIR EWMA control limits (18) have from the center line to the limiting FIR EWMA control limits $\mu_x \pm \rho \times \sigma_x \sqrt{\lambda/(2 - \lambda)}$. Steiner [35] recommends to set $\alpha = 0.3$ and $f = 0.5$ in general.
IV. EWMA Q CONTROL CHARTS

Replacing \( x_k \) in (10) with a Q-Statistic \( Q_k(x_k) \) defined in (1), (2), (4), or (6), results in the statistic

\[
z_k = \lambda Q_k(x_k) + (1 - \lambda)z_{k-1}, \quad k \geq 1, \tag{19}\]

where \( \lambda \in (0,1) \) is the appropriately chosen weighting factor. Since the initial value of \( z_k \) highly impacts the calculation of all subsequent values, \( z_{k-1} \) is set to the starting value of the considered Q-Statistic \( Q_k(x_k) \). For instance, according to Case UU in (6) the initial value of the sequence \( z_k \) is \( z_0 = Q_0(x_0) \).

As mentioned in Section II, the Q-Statistics \( Q_k(x_k) \) are independent and \( \mathcal{N}(0,1) \) distributed. Hence, by (12) \( \mu_k = \mu_{Q_k(x_k)} = 0 \), and in view of (14)

\[
\sigma^2_{Q_k(x_k)} = \frac{\lambda}{2 - \lambda} \left(1 - (1 - \lambda)^{2k}\right). \tag{20}\]

Then the EWMA Q control chart limits are given by

\[
\pm \rho \sqrt{\frac{\lambda}{2 - \lambda} (1 - (1 - \lambda)^{2k})}, \tag{21}\]

compare (16), and the FIR adjusted EWMA Q control chart limits are of the form

\[
\pm \rho \times \text{FIR}_{adj} \times \sqrt{\lambda/(2 - \lambda)} \tag{22}\]

in view of (18).

In order to assure a good EWMA Q control chart performance, \( \lambda = 0.25 \) and \( \rho = 2.998 \) is chosen resulting in EWMA Q control chart limits of \( \pm 1.133 \), or in \( \pm 1.133 \times \text{FIR}_{adj} \). Thus \( |z_k| > 1.133 \) (or, \( |z_k| > 1.133 \times \text{FIR}_{adj} \)) indicates an out-of-control situation of a process under consideration. Figure 4 depicts an EWMA Q control chart according to the EWMA Q-Statistic (19) and the control limits (16) with \( \mu_x = 0 \) and \( \sigma_x = 1 \). Since these control limits are tighter than those in Figure 1 two additional out-of-control events at day 5 and 6 are detected. Note that the non-connected points in Figure 4 represent the EWMA Q-Statistic values.

As mentioned in Section III-B, the sensitivity to detect out-of-control events can be increased at the start-up phase of a process, if the FIR enhancement defined in (17) is taken into account. Figure 5 and Figure 6 visualize the FIR enhancements of Steiner [35] and Haq, Brown and Moltchanova [13], respectively, with \( a = 0.3 \) and \( f = 0.5 \).

In both cases one additional out-of-control event at day 7 is detected.

Analog to Figure 2, Figure 7 illustrates the modified FIR adjusted EWMA Q control chart when the out-of-control event at day 4 in Figure 6 is eliminated automatically.

Fig. 4. An EWMA Q control chart (based on Eq. (16))

Fig. 5. A FIR adjusted EWMA Q control chart (Steiner [35])

Fig. 6. A modified FIR adjusted EWMA Q control chart (Haq et al. [13])

V. Q-STATISTICS PREDICTIONS BASED ON EWMA

Prediction or forecasting is a very important function in many business areas, see e.g. Fildes et al. [9]. The goal of forecasting is to predict values of a time series as reliable as possible in the (near) future. One of the most widely applied forecasting method that continually updates a forecast is exponential smoothing, see, for instance, Fildes et al. [9], Gardner [11], [12], Hyndman and Athanasopoulos [19], or Hyndman et al. [22].

A. Q-Statistics and Simple Exponential Smoothing

Simple exponential smoothing is appropriate for short-term prediction, e.g. for the forecast of the next, one-step-ahead time series value.

The statistic (19) can be rewritten as

\[
z_k = z_{k-1} + \lambda(Q_k(x_k) - z_{k-1}), \quad \lambda \in (0,1), \tag{23}\]

where \( z_{k-1} \) is interpreted as the forecast or prediction of the Q-Statistic \( Q_k(x_k) \) for iteration \( k \). The difference \( Q_k(x_k) - z_{k-1} \) is termed the forecast error at iteration \( k \). Therefore, the error correction form of (23) is equal to

\[
z_k = z_{k-1} + \lambda e_k \quad \text{with} \quad e_k = Q_k(x_k) - z_{k-1}. \tag{24}\]

In order to express that (23) resp. (24) represents a prediction or forecast, an alternative representation of \( z_k \) in (24) is

\[
z_{k+1} = z_k + \lambda e_k \quad \text{with} \quad e_k = Q_k(x_k) - z_k, \quad \lambda \in (0,1). \tag{25}\]
reflecting that a tomorrow’s predicted value \( z_{k+1} \) is equal to the
today’s predicted value \( z_k \) plus the smoothing parameter \( \lambda \) times the
today’s prediction error \( e_k \). This way to calculate EWMA based one-
step-ahead forecasts is termed simple or single exponential smoothing
due to the presence of the single smoothing parameter \( \lambda \) in (25).

The value of the parameter \( \lambda \) has a significant impact on smoothing
the predicted values. Obviously, a (too) small value of \( \lambda \) entails
that the new forecast \( z_{k+1} \) is quite equal to the previous forecast
\( z_k \), a (too) large \( \lambda \) focuses on the prediction error \( e_k \). Thus a natural
question arises: How to determine \( \lambda \) in order to get a reliable forecast
effect? A simple trial and error approach seeks an (approximately)
optimal \( \lambda \) that minimizes the sum of squared prediction errors \( \sum_i e_i^2 \).
This can be accomplished by replacing \( \lambda \) with different values in (25),
by calculating the corresponding sums of squared prediction errors for
each value of \( \lambda \) and finally by selecting that value of \( \lambda \) which
yields the minimum value of the sum. An alternative approach is to
solve the quadratic programming problem \( \min \sum_i e_i^2 \) subject to \( \lambda \) by
means of a mathematical programming software package.

Another alternative approach is to utilize the \texttt{R} Statistical Com-
puting Environment [31], in particular the standard package \texttt{stats},
or the package \texttt{forecast} developed by Hyndman and Khandakar
[20]. Both packages implement the forecast procedure described in
Holt [15] and enhanced by Winters [39] to determine a solution to
(25).

### B. Q-Statistics and Double Exponential Smoothing

In general, time series data exhibit random variations, but in some
cases the data may show a shift to higher or lower values over a certain
time period. In this case a trend pattern exists. A trend reflects the
long-term direction of the considered observations series.

According to the \( Q \) control chart run rules (see Section II-B), a
positive trend in the observations exists if (at least) six consecutive out
of the \( k \) actual observations reveal a monotone increasing pattern, i.e.
if \( Q_{j-5}(x_{j-5}) < Q_{j-4}(x_{j-4}) < \cdots < Q_j(x_j) \) with \( j \leq k, \) see e.g.
Hoyer and Ellis [17, Rule 5]. A negative trend is defined analogously
by a monotone decreasing pattern. In exponential smoothing a trend
is of the form

\[
t_k = \nu (z_k - z_{k-1}) + (1 - \nu)t_{k-1},
\]

whereby \( \nu \in (0, 1) \) is a smoothing parameter. Obviously, the term \( t_k \)
defines the local linear trend of an observation series for each \( k \geq 1 \).

### C. Measuring the Accuracy of Q-Statistics Predictions

Evidently, in order to decide whether an actual prediction calculated
by (25) or by (28) is acceptable or not, some measure of forecast
accuracy is required. A crucial component of some underlying metrics
is the mean of the (absolute or squared) one-step-ahead forecast errors
\( e_j, 1 \leq j \leq k, \) see e.g. Hyndman et al. [22].

For instance, the tracking signal \( TS_k \) is in general defined by

\[
TS_k = \frac{\sum_{j=1}^{k} e_j}{\sum_{j=1}^{k} |e_j|}.
\]

\( TS_k \) sets the bias in relation to the average absolute forecast error
and is recalculated each time a new individual observation has been
made. Ideally, the various tracking signal values \( TS_j, 1 \leq j \leq k, \) fluctuate
around zero within user-defined acceptable control limits, e.g. \( \pm 4 \). If a tracking signal \( TS_j \) exceeds the control limits, then the
prediction error may be nonrandom and the actual prediction is no
longer beneficial. In this case, reset the forecasting and re-start.

The Mean Absolute Scaled Error (MASE) prediction accuracy
measure of Hyndman and Koehler [21] checks the mean of the absolute
forecast errors \( e_j, 1 \leq j \leq k \) against that of the absolute naïve
one step forecast errors \( Q_j(x_j) - Q_{j-1}(x_{j-1}) \) for \( 2 \leq j \leq k - 1 \):

\[
MASE_k = \frac{1}{k-1} \sum_{j=1}^{k} |e_j|.
\]

Clearly, if \( MASE_k < 1 \), then the forecast is better than the average
 naïve forecast, and the forecast is worse than the average naïve
forecast when \( MASE_k > 1 \). In addition, each \( MASE_k \) is undefined
when all differences \( |Q_j(x_j) - Q_{j-1}(x_{j-1})| = 0 \) for \( 2 \leq j \leq k \).

Another forecast accuracy measure was proposed by the Dutch
econometrician Henri Theil who mainly studied the inequality distribu-
tion of income and asset. Related to times series the Theil accuracy
measure \( U_k \) is defined by

\[
U_k = \frac{\sum_{j=1}^{k} e_j^2}{\sum_{j=1}^{k} (Q_j(x_j) - Q_{j-1}(x_{j-1}))^2}.
\]

Like \( MASE_k \) in (30) Theil’s measure \( U_k \) quantifies how well the series
of forecast errors compares to the corresponding series of naïve one
step forecast errors. Hence, if \( U_k < 1 \), then the forecast is better than
the naïve one step forecast. The forecast is worse, when \( U_k > 1 \).

Trigg and Leach [38] modify (29) by replacing the prediction errors
\( e_j \) and \( |e_j| \) with the corresponding smoothed prediction errors
resulting in the smoothed error tracking signal defined by

\[
SETS_k = \frac{\tilde{e}_k}{\tilde{a}_k} = \frac{\gamma e_k + (1 - \gamma)\tilde{e}_{k-1}}{\gamma |\tilde{e}_k| + (1 - \gamma)\tilde{a}_{k-1}}.
\]
Evidently, \( \text{SET}_k \in [-1, +1] \). With respect to Section III, the limiting variance of \( \text{SET}_k \) is equal to \( c \times \gamma/(2 - \gamma) \), where \( c \approx 1.5 \) in general. In order to determine a value of \( \gamma \), Trigg [37], and Trigg and Leach [38] propose to select a value of \( \gamma \in [0.1, 0.3] \). Then the smoothed error tracking signal \( \text{SET}_k \) in (32) indicates an out-of-control forecast, when \( \text{SET}_k \) exceeds one of the corresponding control limits. For instance, when \( \gamma = 0.1 \) the control limits are \( \pm 0.51 \), see Trigg [37]. When \( \text{SET}_k \) is in-control, Trigg and Leach [38] propose to set \( \lambda = |\text{SET}_k| \) in (23) and (28), respectively. When \( \text{SET}_k \) is out-of-control, Trigg [37] recommends to check the forecast in order to detect a potential assignable cause according to the transformed \( Q_k(x_k) \) or to the real observation \( x_k \).

A problem arises from the tracking signal given in (32), when \( e_k \) is a perfect prediction, i.e. when \( e_k = 0 \). In this case, \( \text{SET}_k = \text{SET}_{k-1} \). In view of (30), this problem is solved when the denominator in (32) is replaced by a simple exponentially smoothing expression \( q_k \), which consists of the difference between the actual observation \( Q_k(x_k) \) and its predecessor \( Q_{k-1}(x_{k-1}) \), i.e. \( q_k = Q_k(x_k) - Q_{k-1}(x_{k-1}) \). Then the new smoothed error tracking signal is

\[
\text{SET}_k = \frac{\hat{e}_k}{q_k} = \frac{\gamma e_k + (1 - \gamma) \hat{e}_{k-1}}{\gamma q_k + (1 - \gamma) q_{k-1}},
\]

(33)

compare (30). In order to determine a value of \( \gamma \) one can proceed by following the approach described in the previous paragraph. Or, simulations or practical experiments have to be conducted, which will be done by the authors in the near future.

When utilizing the \( R \) package forecast written by Hyndman and Khandakar [20] a first insight into the performance of the various tracking signals is given. In the next section a practical example is investigated by applying the forecast package.

VI. AN EXAMPLE

The results of Section IV indicate that EWMA \( Q \) control charts are attractive to control and to monitor a Software Development Process (SDP) and its various phases or stages. In Section V forecasts based on EWMA \( Q \) control charts and accuracy measures of such forecasts were introduced. This section exemplifies that forecasts are beneficial to control and to monitor an SDP, in this case the aggregated forecasts were introduced. This section exemplifies that forecasts are based on EWMA \( Q \) control charts and accuracy measures of such forecasts.

In this paper the forecast horizon \( h \) is always set to \( h = 1 \), since it is a beneficial test strategy to directly react to out-of-control events.

In order to apply the \( \text{ses} \) and \( \text{holt} \) functions to the number of defects the corresponding \( Q \)-Statistic values are calculated, see Table I. Recall that in view of (6), \( Q_1(x_1) \) and \( Q_2(x_2) \) are not defined.

TABLE I. \( Q \)-STATISTIC VALUES

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( Q_1(x_1) )</th>
<th>( CI_{90} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1]</td>
<td>NA</td>
<td>0.1481747</td>
</tr>
<tr>
<td>[6]</td>
<td>-0.444591</td>
<td>0.459483</td>
</tr>
<tr>
<td>[11]</td>
<td>-0.6677563</td>
<td>0.4852984</td>
</tr>
<tr>
<td>[16]</td>
<td>0.2133119</td>
<td>0.2073223</td>
</tr>
<tr>
<td>[21]</td>
<td>-0.8308747</td>
<td>0.581</td>
</tr>
</tbody>
</table>

In a first step, the functions \( \text{ses} \) and \( \text{holt} \) are applied to predict the \( Q \)-Statistic value \( Q_{10}(x_{10}) \). The results are listed in Table II.

TABLE II. PREDICTING \( Q_{10}(x_{10}) \)

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( Q_{10}(x_{10}) )</th>
<th>80% CI</th>
<th>95% CI</th>
<th>MASE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{ses} )</td>
<td>0.381</td>
<td>0.688</td>
<td>-0.66, 1.82</td>
<td>-1.31, 2.48</td>
</tr>
<tr>
<td>( \text{holt} )</td>
<td>-0.096</td>
<td>0.688</td>
<td>-1.26, 1.07</td>
<td>-1.87, 1.68</td>
</tr>
</tbody>
</table>

Since \( MASE < 1 \) in both cases, the accuracy of the forecasts are better than those of the average naive forecasts. Furthermore, a closer look at the summary function of the \( \text{ses} \) and \( \text{holt} \) forecasts reveals that a trend cannot be detected.

When predicting \( Q_{18}(x_{18}) \) the \( \text{holt} \) function detects a trend defined in (26). The trend parameter \( \nu \) is equal to 0.1222. The results of the \( \text{ses} \) and \( \text{holt} \) forecasts of \( Q_{18}(x_{18}) \) are listed in Table III.
Obviously, the accuracy of both forecasts is not better than that of the average naïve forecast.

$Q$-Statistic forecasts can be controlled and monitored by means of control charts. The associated control limits can be calculated by taking into account the standard deviation of the forecast errors $e_k$. The summary function of the ses and holt procedures lists the standard deviation. The control limits are in general $±2$ or $±3$ times the standard deviation. The $Q$-Statistic forecasts listed in Table II and in Table III do not reveal an out-of-control forecast.

In view of (6) a forecast of the data point $x_k$ can readily be determined:

$$x_k = \hat{x}_{k-1} + \sqrt{\frac{k}{k-1}} \times s_{k-1} \times G_{k-2}^{-1} \{ \Phi \left[ Q_k(x_k) \right] \}, \quad (34)$$

where $G_{k-2}^{-1}$ denotes the inverse of the Student $t$ cumulative distribution function with $k − 2$ degrees of freedom, and $Φ$ the standard normal cumulative distribution function. Thus, replacing the term $Q_k(x_k)$ in (34) with a corresponding $Q$-Statistic forecast results in a forecast of $x_k$. A forecast of the range of $x_k$ is calculated by replacing the term $Q_k(x_k)$ with the endpoints of the corresponding confidence interval of $80\%$ or $90\%$ confidence level. For instance, in view of the values listed in Table III one obtains:

**TABLE III. PREDICTING $Q_{18}(x_{18})$**

<table>
<thead>
<tr>
<th>Inc.</th>
<th>forecast $Q_{18}(x_{18})$</th>
<th>80% CI</th>
<th>95% CI</th>
<th>MAPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ses</td>
<td>0.348</td>
<td>2.078</td>
<td>$(-0.64, 1.33)$</td>
<td>$(-1.10, 1.85)$</td>
</tr>
<tr>
<td>holt</td>
<td>0.182</td>
<td>2.078</td>
<td>$(-0.86, 1.22)$</td>
<td>$(-1.41, 1.77)$</td>
</tr>
</tbody>
</table>

Suppose that the forecast $x_k$ has been calculated and the "true" $x_k$ has been observed. If the true $x_k$ does not lie in one of the confidence intervals of the forecast $x_k$, then an investigation why this event occurred is appropriate in order to potentially decrease the number of aggregated defects immediately.

**TABLE IV. PREDICTING EVENT $x_{18}$**

<table>
<thead>
<tr>
<th>Inc.</th>
<th>forecast $x_{18}$</th>
<th>$x_{18}$</th>
<th>80% CI</th>
<th>95% CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>ses</td>
<td>24</td>
<td>36</td>
<td>$[17, 31]$</td>
<td>$[14, 35]$</td>
</tr>
<tr>
<td>holt</td>
<td>23</td>
<td>36</td>
<td>$[16, 30]$</td>
<td>$[12, 35]$</td>
</tr>
</tbody>
</table>

**VII. CONCLUSION**

Traditional Shewhart control charts are a successfully applied Statistical Process Control (SPC) tool for controlling and monitoring the variation of long-run mass production processes. To ascertain robust and valid control chart limits, a sufficiently large set of observations drawn from a process under consideration is required. Hence, Shewhart control charts cannot readily be adopted to the Software Development Process (SDP), since the SDP does not provide in general the needed large set of observations. As indicated in this paper, $Q$ control charts are a highly appropriate alternative to Shewhart control charts in the SDP context, because they enable early detection of nonrandom process behavior and the controlling and monitoring of the SDP as a short-run process in real time. These $Q$ control chart properties originate from pre-defined, constant control limits making the Phase I as part of classic Shewhart control charts in order to stabilize the control chart limits redundant.

This paper focused on the performance of $Q$-Statistics based EWMA control charts, inclusive the FIR adjustment enhancement, and their forecasting feature and on the quality of the forecasts. Although only limited experiments have been conducted, the results are promising. So further practical experiments have to be conducted in order to confirm the findings obtained so far.

**REFERENCES**


